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Energy norm *a posteriori* error estimation for divergence-free discontinuous Galerkin approximations of the Navier–Stokes equations

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SUMMARY

We develop the energy norm *a posteriori* error analysis of exactly divergence-free discontinuous RT_k/Q_k Galerkin methods for the incompressible Navier–Stokes equations with small data. We derive upper and local lower bounds for the velocity–pressure error measured in terms of the natural energy norm of the discretization. Numerical examples illustrate the performance of the error estimator within an adaptive refinement strategy. Copyright © 2008 John Wiley & Sons, Ltd.

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1. INTRODUCTION

In this paper, we derive a residual-based energy norm *a posteriori* error estimator for exactly divergence-free discontinuous Galerkin (DG) methods for the incompressible Navier–Stokes equations

$$\begin{aligned} -\nu\Delta\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p=\mathbf{f} &\quad \text{in } \Omega\subset\mathbb{R}^2 \\ \nabla\cdot\mathbf{u}=0 &\quad \text{in } \Omega \\ \mathbf{u}=\mathbf{0} &\quad \text{on } \Gamma=\partial\Omega \end{aligned} \tag{1}$$

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Here, $\nu>0$ is the kinematic viscosity, $\mathbf{u}$ the velocity, p the pressure, and $\mathbf{f}\in L^2(\Omega)^2$ an external body force. The domain Ω is assumed to be a Lipschitz polygon in $\mathbb{R}^2$. Throughout, we assume that $\nu^{-2}\|\mathbf{f}\|_{L^2(\Omega)}$ is sufficiently small. Then the Navier–Stokes system has a unique solution $(\mathbf{u},p)\in H_0^1(\Omega)^2\times L_0^2(\Omega)$, where $H_0^1(\Omega)^2$ is the usual vector-valued Sobolev space with zero boundary values and $L_0^2(\Omega)$ is the space of square integrable functions with vanishing mean value. Moreover, we have the stability bound

$$\|\nabla\mathbf{u}\|_{L^2(\Omega)}\leqslant C_P\nu^{-1}\|\mathbf{f}\|_{L^2(\Omega)} \tag{2}$$

with $C_P>0$ denoting the Poincaré constant of Ω.

Exactly divergence-free DG methods for the incompressible Navier–Stokes equations (1) were recently introduced in [1]. They are based on divergence-conforming finite element spaces for the approximation of the velocity, such as Brezzi–Douglas–Marini (BDM) or Raviart–Thomas (RT) spaces, and on matching discontinuous spaces for the approximation of the pressure. The H^1-conformity of the velocity approximation is then enforced weakly using the DG approach. The resulting methods have been shown to be locally conservative, inf–sup stable, and optimally convergent. They were inspired by the use of BDM elements for the analysis of DG schemes in [2]. In the lowest order case, they have been shown to be closely related to the well-known marker and cell (MAC) scheme; cf. [3]. The use of element pairs beyond BDM and RT is discussed in [4]. In this article, we will focus on RT elements but remark here that the analysis applies in exactly the same way to BDM elements.

The exactly divergence-free methods in [1] originated from the ideas introduced in [5]. There, an element-by-element post-processing procedure was devised to render DG velocity approximations exactly divergence free. The divergence-conforming velocity approximations in [1] can then be understood and analyzed in the setting of [5], with a post-processing operator that is equal to the identity operator. The work [5], in turn, builds upon the earlier papers [6–8], where DG methods for linear incompressible flow problems were proposed and analyzed.

In this paper, we develop the energy norm *a posteriori* error estimation for the RT_k/Q_k method proposed in [1] for quadrilateral meshes. Here, the approximation of the velocity is based on RT elements of order k, whereas the pressure is discretized using tensor product polynomials of order k. We derive a residual-based error estimator that is both reliable and efficient for the error measured in a natural energy norm that includes the broken H^1-norm for the error in the velocity and the L^2-norm for the error in the pressure. Our technique of proof relies on the approach introduced in [9] for the Stokes problem. Let us also mention that *a posteriori* error analyses for DG methods applied to elliptic problems can be found, e.g. in [10–16] and the references therein.

The outline of this paper is as follows. In Section 2, we describe the RT_k/Q_k methods from [1] and establish the stability properties that are crucial for our analysis. For simplicity, we restrict ourselves to the interior penalty approach for the discretization of the diffusive terms. In Section 3, we present our energy norm error estimator and state our main result. The proof of these results is carried out in Section 4. In Section 5, we present a series of numerical experiments that show the usefulness of our estimator in adaptive refinement strategies. Finally, we conclude our presentation in Section 6.

2. DIVERGENCE-FREE DG METHODS

In this section, we recall the RT_k/Q_k–DG method for the Navier–Stokes equations. We follow [1, 5]. However, instead of the local discontinuous Galerkin (LDG) approach presented

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there, we employ the interior penalty (IP) method for the discretization of the Laplace operator.

2.1. Discretization

We assume that the domain Ω can be partitioned into shape-regular rectangular meshes $\mathscr{T}_h = \{K\}$. The diameter of an element K is denoted by h_K, and the mesh size parameter h of $\mathscr{T}_h$ is the piecewise constant function h with $h|_K = h_K$. We will assume that each edge of a mesh cell K is a boundary edge, an edge of a neighboring cell, or consists of two equally long edges of refined neighboring cells. The latter corresponds to the so-called one-irregular meshes with one 'hanging node' on refined edges. We further assume that the neighboring nodes of hanging nodes are not hanging themselves. The adaptively generated meshes in our numerical experiments satisfy these properties, see [17].

Remark 2.1
The inf–sup stability of discretizations with hanging nodes using RT elements is in part still an open question. For quadrilaterals with one-irregular meshes, a stability proof exists only for the pair RT_k/Q_k defined in (3) below with $k \geqslant 2$; see [18]. Nevertheless, we conjecture from our computational results that stability also holds for $k = 1$. For triangles with hanging nodes, there is no stability result for the divergence-free elements proposed in [1]; on the other hand, locally refined triangular meshes without hanging nodes can be obtained using bisection. The results below are all to be read in view of the restrictions cited in this remark.

We will use the standard average and jump operators. To define them, we denote by $\mathscr{E}(\mathscr{T}_h)$ the set of all edges in $\mathscr{T}_h$, by $\mathscr{E}_I(\mathscr{T}_h)$ the set of all interior edges, and by $\mathscr{E}_\mathrm{B}(\mathscr{T}_h)$ the set of all boundary edges. The length of an edge E is denoted by h_E.

Let now K^+ and K^- be two adjacent elements of $\mathscr{T}_h$ that share an interior edge $E = \partial K^+ \cap \partial K^- \in \mathscr{E}_I(\mathscr{T}_h)$. Let φ be any piecewise smooth function (matrix, vector, or scalar valued), and let us denote the traces of φ on E taken from within the interior of $K^\pm$ by $\varphi^\pm$. We then define the average operator $\{\!\{\cdot\}\!\}$ across the edge E as

$$\{\!\{\varphi\}\!\} = \tfrac{1}{2}(\varphi^+ + \varphi^-)$$

Further, let $\underline{\sigma}$, $\mathbf{u}$, and p be a piecewise smooth matrix-valued, vector-valued and scalar-valued functions, respectively. The jumps $[\![\cdot]\!]$ of these functions across E are defined as

$$[\![\underline{\sigma}\mathbf{n}]\!] = \underline{\sigma}^+\mathbf{n}^+ + \underline{\sigma}^-\mathbf{n}^-, \quad [\![\mathbf{u}\otimes\mathbf{n}]\!] = \mathbf{u}^+\otimes\mathbf{n}^+ + \mathbf{u}^-\otimes\mathbf{n}^-, \quad [\![p\mathbf{n}]\!] = p^+\mathbf{n}^+ + p^-\mathbf{n}^-$$

where $\mathbf{n}^\pm$ denote the outward unit normal vectors on the boundary $\partial K^\pm$ of the elements $K^\pm$. Analogously, for a boundary edge $E \in \mathscr{E}_\mathrm{B}(\mathscr{T}_h)$, we set $\{\!\{\varphi\}\!\} = \varphi$, $[\![\underline{\sigma}\mathbf{n}]\!] = \underline{\sigma}\mathbf{n}$, $[\![\mathbf{u}\otimes\mathbf{n}]\!] = \mathbf{u}\otimes\mathbf{n}$, and $[\![p\mathbf{n}]\!] = p\mathbf{n}$. Here, $\mathbf{n}$ is the outward unit normal on Γ. Finally, if the jump appears quadratically, we abbreviate $[\![\mathbf{u}]\!] = \mathbf{u}^+ - \mathbf{u}^-$.

For an approximation order $k \geqslant 1$, we then seek DG approximations to the Navier–Stokes equations in the finite element space $\mathbf{V}_h \times Q_h$, where

$$\begin{aligned} \mathbf{V}_h &= \{\mathbf{v} \in H(\mathrm{div};\Omega) : \mathbf{v}|_K \in \mathrm{RT}_k(K), K \in \mathscr{T}_h, \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \Gamma\} \\ Q_h &= \{q \in L_0^2(\Omega) : q|_K \in Q_k(K), K \in \mathscr{T}_h\} \end{aligned} \tag{3}$$

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with $Q_k(K)$ denoting the tensor product polynomials of degree k on K, $\mathrm{RT}_k(K) \supset Q_k(K)^2$ the RT polynomials of order k on K, see, e.g. [19] and the references therein, and

$$H(\mathrm{div};\Omega) = \{\mathbf{v} \in L^2(\Omega)^2 : \nabla\cdot\mathbf{v} \in L^2(\Omega)\}$$

A crucial property of the pair $\mathbf{V}_h \times Q_h$ is that, on the meshes considered,

$$\nabla\cdot\mathbf{V}_h \subset Q_h \tag{4}$$

see the discussion in [1, Section 2.2.2].

We consider the DG method: find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$\begin{aligned} A_h(\mathbf{u}_h,\mathbf{v}) + O_h(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v}) + B_h(\mathbf{v},p_h) &= \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} \\ B_h(\mathbf{u}_h,q) &= 0 \end{aligned} \tag{5}$$

for all $(\mathbf{v},q) \in \mathbf{V}_h \times Q_h$.

The forms A_h, O_h, and B_h are associated with the discretizations of the Laplacian, the convective term, and the incompressibility condition, respectively.

The form B_h is given by

$$B_h(\mathbf{v},q) = -\int_\Omega \nabla\cdot\mathbf{v} q\,\mathrm{d}\mathbf{x}$$

From the second equation in (5), it then follows that $\nabla\cdot\mathbf{u}_h$ is orthogonal to all pressures in Q_h. Hence, inclusion (4) implies that $\mathbf{u}_h$ is indeed exactly divergence free.

For the form A_h, we choose the classical (symmetric) IP discretization defined by

$$\begin{aligned} A_h(\mathbf{u},\mathbf{v}) = &\sum_{K\in\mathcal{T}_h}\int_K \nu\nabla\mathbf{u}:\nabla\mathbf{v}\,\mathrm{d}\mathbf{x} - \sum_{E\in\mathcal{E}(\mathcal{T}_h)}\int_E [\![\mathbf{v}\otimes\mathbf{n}]\!]:\{\!\{\nu\nabla u\}\!\}\,\mathrm{d}s \\ &- \sum_{E\in\mathcal{E}(\mathcal{T}_h)}\int_E [\![\mathbf{u}\otimes\mathbf{n}]\!]:\{\!\{\nu\nabla\mathbf{v}\}\!\}\,\mathrm{d}s + \sum_{E\in\mathcal{E}(\mathcal{T}_h)}\kappa_E\nu\int_E [\![\mathbf{u}]\!]\cdot[\![\mathbf{v}]\!]\,\mathrm{d}s \end{aligned}$$

The parameter κ_E is the IP parameter stabilizing the DG form. In order to ensure the coercivity of the form A_h, it has to be chosen sufficiently large. For a boundary edge $E \in \mathcal{E}_\mathrm{B}(\mathcal{T}_h)$ of a mesh cell K, its lower limit can be determined by an inverse estimate and is of the form $s/h_\perp$, where $h_\perp$ is the length of K orthogonal to E and s depends on the polynomial degree and the shape of the cell. In particular, for rectangular mesh cells, stability is obtained for

$$\kappa_E > \frac{k(k+1)}{2h_\perp}$$

and we usually choose twice the value in our numerical tests. On interior edges, stability is obtained by taking $\frac{1}{2}$ of the mean value of this value from both cells, respectively. We will always assume that this parameter is chosen such that stability is guaranteed.

We point that, for the discretization of the Laplace operator, several other DG methods can be chosen, for which our results hold true as well; see the discussions in [1, 18, 20, Table I]. For these methods, we at least require stability and consistency; for our purposes, we considered the LDG and (symmetric) IP methods.

Finally, to define the convective form O_h, let $\mathbf{w}$ be a piecewise smooth and divergence-free flow field in the space

$$\mathbf{J}(\mathscr{T}_h)=\{\mathbf{v}\in H(\mathrm{div};\Omega):\nabla\cdot\mathbf{v}\equiv 0 \text{ in } \Omega, \mathbf{v}|_K\in H^1(K)^2, K\in\mathscr{T}_h\}$$

Clearly, the discrete velocity field $\mathbf{u}_h$ belongs to this space. We then take O_h to be the standard upwind form introduced in [21, 22]

$$O_h(\mathbf{w};\mathbf{u},\mathbf{v})=-\sum_{K\in\mathscr{T}_h}\int_K(\mathbf{w}\cdot\nabla)\mathbf{v}\cdot\mathbf{u}\,\mathrm{d}\mathbf{x}+\sum_{K\in\mathscr{T}_h}\int_{\partial K}\left[\mathbf{w}\cdot\mathbf{n}_K\{\!\{\mathbf{u}\}\!\}-\frac{1}{2}|\mathbf{w}\cdot\mathbf{n}_K|(\mathbf{u}^{\mathrm{e}}-\mathbf{u})\right]\cdot\mathbf{v}\,\mathrm{d}s$$

Here, $\mathbf{u}^{\mathrm{e}}$ denotes the exterior trace of $\mathbf{u}$ taken over the edge under consideration and set to zero on the boundary. Upon integration by parts, we also have

$$O_h(\mathbf{w};\mathbf{u},\mathbf{v})=\sum_{K\in\mathscr{T}_h}\int_K(\mathbf{w}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}+\sum_{K\in\mathscr{T}_h}\int_{\partial K}\left[\frac{1}{2}\mathbf{w}\cdot\mathbf{n}_K(\mathbf{u}^{\mathrm{e}}-\mathbf{u})-\frac{1}{2}|\mathbf{w}\cdot\mathbf{n}_K|(\mathbf{u}^{\mathrm{e}}-\mathbf{u})\right]\cdot\mathbf{v}\,\mathrm{d}s$$

This completes the definition of the DG methods for the incompressible Navier–Stokes equations. We point out that, as $\mathbf{u}_h$ is exactly divergence free, the resulting DG methods are locally conservative and energy stable, cf. [1, 5].

2.2. *Stability properties*

In this section, we recapitulate the main stability properties of the DG forms from [1, 5, 18].

First, we rewrite the IP form A_h in the form

$$A_h(\mathbf{u},\mathbf{v})=\int_\Omega v\nabla\mathbf{u}:\nabla\mathbf{v}\,\mathrm{d}\mathbf{x}-\int_\Omega v\nabla\mathbf{u}:\underline{\mathscr{L}}(\mathbf{v})\,\mathrm{d}\mathbf{x}-\int_\Omega v\nabla\mathbf{v}:\underline{\mathscr{L}}(\mathbf{u})\,\mathrm{d}\mathbf{x}+\sum_{E\in\mathscr{E}(\mathscr{T}_h)}\kappa_E v\int_E[\![\mathbf{u}]\!]\cdot[\![\mathbf{v}]\!]\,\mathrm{d}s$$

where the lifting operator $\underline{\mathscr{L}}:\mathbf{V}_h\to\underline{\Sigma}_h$ is given by

$$\int_\Omega\underline{\mathscr{L}}(\mathbf{v}):\underline{\tau}\,\mathrm{d}\mathbf{x}=\sum_{E\in\mathscr{E}(\mathscr{T}_h)}\int_E[\![\mathbf{v}\otimes\mathbf{n}]\!]:\{\!\{\underline{\tau}\}\!\}\,\mathrm{d}s\quad\forall\underline{\tau}\in\underline{\Sigma}_h$$

with

$$\underline{\Sigma}_h=\{\underline{\tau}\in L^2(\Omega)^{2\times 2}:\underline{\tau}|_K\in Q_{k+1}(K)^{2\times 2}, K\in\mathscr{T}_h\}$$

We can now easily extend the form A_h to the space $\mathbf{V}(h)=H_0^1(\Omega)+\mathbf{V}_h$ using the same definition of the lifting operator. This space is endowed with the broken H^1-norm

$$\|\mathbf{u}\|_{1,h}^2=\sum_{K\in\mathscr{T}_h}\|\nabla\mathbf{u}\|_{L^2(K)}^2+\sum_{E\in\mathscr{E}(\mathscr{T}_h)}\kappa_E v\|[\![\mathbf{u}]\!]\|_{L^2(E)}^2$$

Then, the form A_h satisfies the following continuity and coercivity properties: If the IP parameter is chosen as above, then there are constants $c_a>0$ and $\alpha>0$, independent of the viscosity and the mesh size, such that

$$A_h(\mathbf{u},\mathbf{v})\leqslant v c_a\|\mathbf{u}\|_{1,h}\|\mathbf{v}\|_{1,h},\quad \mathbf{u},\mathbf{v}\in\mathbf{V}(h) \tag{6}$$

$$A_h(\mathbf{u},\mathbf{u})\geqslant v\alpha\|\mathbf{u}\|_{1,h}^2,\quad \mathbf{u}\in\mathbf{V}_h \tag{7}$$

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The form O_h is Lipschitz continuous: There is a constant $c_0>0$, independent of the viscosity and the mesh size, such that

$$|O_h(\mathbf{w}_1;\mathbf{u},\mathbf{v})-O_h(\mathbf{w}_2;\mathbf{u},\mathbf{v})|\leqslant c_0\|\mathbf{w}_1-\mathbf{w}_2\|_{1,h}\|\mathbf{u}\|_{1,h}\|\mathbf{v}\|_{1,h} \tag{8}$$

for all $\mathbf{w}_1,\mathbf{w}_2\in\mathbf{J}(\mathscr{T}_h)$, $\mathbf{u}\in\mathbf{V}(h)$, and $\mathbf{v}\in\mathbf{V}(h)$. This statement has been proven in [5, Proposition 4.2] for $\mathbf{v}\in\mathbf{V}_h$. Using similar arguments, it can be readily seen that it also holds for $\mathbf{v}\in H_0^1(\Omega)^2$. Moreover, there holds

$$O_h(\mathbf{w};\mathbf{u},\mathbf{u})\geqslant 0 \tag{9}$$

for $\mathbf{w}\in\mathbf{J}(\mathscr{T}_h)$ and $\mathbf{u}\in\mathbf{V}_h$ or $\mathbf{u}\in H_0^1(\Omega)^2$.

Finally, the form B_h is continuous and satisfies the discrete inf–sup condition: There exist constants $c_b>0$ and $\beta>0$, independent of the viscosity and the mesh size, such that

$$|B_h(\mathbf{u},p)|\leqslant c_b\|u\|_{1,h}\|p\|_{L^2(\Omega)},\quad \mathbf{u}\in\mathbf{V}(h),\quad p\in L^2(\Omega) \tag{10}$$

$$\sup_{\mathbf{u}\in\mathbf{V}_h}\frac{B_h(\mathbf{u},p)}{\|\mathbf{u}\|_{1,h}}\geqslant\beta\|p\|_{L^2(\Omega)},\quad p\in Q_h \tag{11}$$

While the continuity of B_h is obvious, the inf–sup condition follows from the results in [18]. It holds on regular meshes for any k. For $k\geqslant 2$, it holds on the irregular meshes considered in this paper. As pointed out in Remark 2.1, the stability on irregular meshes for $k=0,1$ remains an open problem. We further refer the reader to [2] for the stability of BDM elements on conforming triangular meshes.

With these stability properties at hand and by proceeding as in the analysis presented in [1, Theorem 3.1] for triangular elements, we readily obtain the following result: If $\nu^{-2}\|\mathbf{f}\|_{L^2(\Omega)}$ is sufficiently small, then the DG discretization (5) has a unique solution $(\mathbf{u}_h,p_h)\in\mathbf{V}_h\times Q_h$ and we have the *a priori* error estimate

$$\|\mathbf{u}-\mathbf{u}_h\|_{1,h}+\|p-p_h\|_{L^2(\Omega)}\leqslant C(\|h^k\mathbf{u}\|_{H^{k+1}(\mathscr{T}_h)}+\|h^k p\|_{H^k(\mathscr{T}_h)})$$

with a constant $C>0$ independent of the mesh size. Here, the $\|.\|_{H^k(\mathscr{T}_h)}$ denotes the cellwise H^k-norm. For later use, we also note that the discrete velocity $\mathbf{u}_h$ satisfies the bound

$$\|\mathbf{u}_h\|_{1,h}\leqslant c_p\nu^{-1}\|\mathbf{f}\|_{L^2(\Omega)} \tag{12}$$

where c_p is the constant in the discrete Poicaré inequality: There is $c_p>0$ independent of the mesh size such that

$$\|\mathbf{v}\|_{L^2(\Omega)}\leqslant c_p\|\mathbf{v}\|_{1,h}\quad\forall\mathbf{v}\in\mathbf{V}_h \tag{13}$$

see [23, Lemma 2.1; 24].

2.3. Additional stability properties

In this section, we establish two additional stability results that are key in our *a posteriori* error analysis. To that end, we introduce the global form

$$\mathscr{A}_h(\mathbf{w})(\mathbf{u},p;\mathbf{v},q)=A_h(\mathbf{u},\mathbf{v})+O_h(\mathbf{w};\mathbf{u},\mathbf{v})+B_h(\mathbf{v},p)-B_h(\mathbf{u},q)$$

for any $\mathbf{w}\in\mathbf{J}(\mathscr{T}_h)$, $(\mathbf{u}, p)$, and $(\mathbf{v}, q)$ in $\mathbf{V}(h)\times L^2(\Omega)$. We then rewrite the DG methods in the form: Find $(\mathbf{u}_h, p_h)\in\mathbf{V}_h\times Q_h$ such that

$$\mathscr{A}_h(\mathbf{u}_h)(\mathbf{u}_h, p_h; \mathbf{v}, q) = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} \quad \forall(\mathbf{v}, q)\in\mathbf{V}_h\times Q_h \tag{14}$$

We further introduce the product norm

$$|||(\mathbf{u}, p)|||^2 = \nu\|\mathbf{u}\|_{1,h}^2 + \nu^{-1}\|p\|_{L^2(\Omega)}^2$$

and define

$$\bar{c}_p = \max\{C_S, c_p\alpha^{-1}\}$$

In the following, we consider small data and assume that

$$c_0\bar{c}_p\nu^{-2}\|\mathbf{f}\|_{L^2(\Omega)} < 1 \tag{15}$$

Then the following continuity and stability properties hold.

Lemma 2.1
Assume (15) and let $\mathbf{w}\in\mathbf{J}(\mathscr{T}_h)$ satisfy

$$\|\mathbf{w}\|_{1,h} \leqslant \bar{c}_p\nu^{-1}\|\mathbf{f}\|_{L^2(\Omega)}$$

Then there is a continuity constant $c_{\mathscr{A}}>0$, only depending c_a and c_b, such that

$$|\mathscr{A}_h(\mathbf{w})(\mathbf{u}, p; \mathbf{v}, q)| \leqslant c_{\mathscr{A}}|||(\mathbf{u}, p)|||\,|||(\mathbf{v}, q)|||$$

for all $\mathbf{u}, \mathbf{v}\in\mathbf{V}(h)$ and $p, q\in L^2(\Omega)$.

Proof
The assertion follows from the continuity of A_h and B_h, the Cauchy–Schwarz inequality and the fact that

$$\begin{aligned}|O_h(\mathbf{w}; \mathbf{u}, \mathbf{v})| &\leqslant c_0\|\mathbf{w}\|_{1,h}\|\mathbf{u}\|_{1,h}\|\mathbf{v}\|_{1,h}\\ &\leqslant c_0\bar{c}_p\nu^{-2}\|\mathbf{f}\|_{L^2(\Omega)}\nu^{1/2}\|\mathbf{u}\|_{1,h}\nu^{1/2}\|\mathbf{v}\|_{1,h} \leqslant \nu^{1/2}\|\mathbf{u}\|_{1,h}\nu^{1/2}\|\mathbf{v}\|_{1,h}\end{aligned}$$

which is due to (8), the bound on $\mathbf{w}$ and the smallness assumption (15). □

Lemma 2.2
Assume (15) and let $\mathbf{w}\in\mathbf{J}(\mathscr{T}_h)$ satisfy

$$\|\mathbf{w}\|_{1,h} \leqslant \bar{c}_p\nu^{-1}\|\mathbf{f}\|_{L^2(\Omega)}$$

Then there is constant $\gamma>0$, only depending on Ω, such that for any tuple $(\mathbf{u}, p)\in H_0^1(\Omega)^2\times L_0^2(\Omega)$ there is $(\mathbf{v}, q)\in H_0^1(\Omega)^2\times L_0^2(\Omega)$ with $|||(\mathbf{v}, q)|||\leqslant 1$ and

$$\mathscr{A}_h(\mathbf{w})(\mathbf{u}, p; \mathbf{v}, q) \geqslant \gamma|||(\mathbf{u}, p)|||$$

Proof
Fix $(\mathbf{u}, p)\in H_0^1(\Omega)^2\times L_0^2(\Omega)$. We have

$$\mathscr{A}_h(\mathbf{w})(\mathbf{u}, p; \mathbf{u}, p) = A_h(\mathbf{u}, \mathbf{u}) + O_h(\mathbf{w}; \mathbf{u}, \mathbf{u}) \geqslant \nu\|\mathbf{u}\|_{1,h}^2 \tag{16}$$

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Here, we have used (9) and that, for $\mathbf{u}\in H_0^1(\Omega)^2$,

$$A_h(\mathbf{u},\mathbf{u}) = \nu\|\nabla\mathbf{u}\|^2_{L^2(\Omega)} = \nu\|\mathbf{u}\|^2_{1,h}$$

Further, due to the continuous inf–sup condition, see [19, 25], there is a field $\bar{\mathbf{v}}\in H_0^1(\Omega)^2$ that satisfies

$$B_h(\bar{\mathbf{v}}, p) \geqslant C_\Omega \nu^{-1}\|p\|^2_{L^2(\Omega)}, \quad \nu^{1/2}\|\bar{\mathbf{v}}\|_{1,h} \leqslant \nu^{-1/2}\|p\|_{L^2(\Omega)}$$

with $C_\Omega>0$ denoting the continuous inf–sup constant, which only depends on Ω. Then,

$$\begin{aligned}\mathscr{A}_h(\mathbf{w})(\mathbf{u}, p; \bar{\mathbf{v}}, 0) &= A_h(\mathbf{u},\bar{\mathbf{v}}) + O_h(\mathbf{w};\mathbf{u},\bar{\mathbf{v}}) + B_h(\bar{\mathbf{v}}, p)\\ &\geqslant C_\Omega\nu^{-1}\|p\|^2_{L^2(\Omega)} - |A_h(\mathbf{u},\bar{\mathbf{v}})| - |O_h(\mathbf{w};\mathbf{u},\bar{\mathbf{v}})|\end{aligned}$$

The H^1-conformity of the functions $\mathbf{u},\bar{\mathbf{v}}$ and the bound for $\bar{\mathbf{v}}$ yield

$$|A_h(\mathbf{u},\bar{\mathbf{v}})| \leqslant \nu\|\mathbf{u}\|_{1,h}\|\bar{\mathbf{v}}\|_{1,h} \leqslant \|\mathbf{u}\|_{1,h}\|p\|_{L^2(\Omega)}$$

Furthermore, employing the continuity of O_h, the bounds for $\mathbf{w}$ and $\bar{\mathbf{v}}$ and the smallness assumption (15), we obtain

$$|O_h(\mathbf{w};\mathbf{u},\bar{\mathbf{v}})| \leqslant c_0\nu^{-1}\|\mathbf{w}\|_{1,h}\|\mathbf{u}\|_{1,h}\|p\|_{L^2(\Omega)} \leqslant \|\mathbf{u}\|_{1,h}\|p\|_{L^2(\Omega)}$$

Hence, the inequality $|ab|\leqslant(\varepsilon/2)a^2+(1/2\varepsilon)b^2$ now gives

$$\begin{aligned}\mathscr{A}_h(\mathbf{w})(\mathbf{u}, p; \bar{\mathbf{v}}, 0) &\geqslant C_\Omega\nu^{-1}\|p\|^2_{L^2(\Omega)} - 2\|\mathbf{u}\|_{1,h}\|p\|_{L^2(\Omega)}\\ &\geqslant \left(C_\Omega - \frac{1}{\varepsilon}\right)\nu^{-1}\|p\|^2_{L^2(\Omega)} - \nu\varepsilon\|\mathbf{u}\|^2_{1,h}\end{aligned} \tag{17}$$

for any $\varepsilon>0$.

From (16) and (17), we conclude that, for $\delta>0$,

$$\begin{aligned}\mathscr{A}_h(\mathbf{w})(\mathbf{u}, p; \mathbf{u}+\delta\bar{\mathbf{v}}, p) &= \mathscr{A}_h(\mathbf{w})(\mathbf{u}, p; \mathbf{u}, p) + \delta\mathscr{A}_h(\mathbf{w})(\mathbf{u}, p; \bar{\mathbf{v}}, 0)\\ &\geqslant (1-\varepsilon\delta)\nu\|\mathbf{u}\|^2_{1,h} + \delta\left(C_\Omega - \frac{1}{\varepsilon}\right)\nu^{-1}\|p\|^2_{L^2(\Omega)}\end{aligned}$$

Taking $\varepsilon = 2/C_\Omega$ and $\delta = C_\Omega/4$, we obtain that

$$\mathscr{A}_h(\mathbf{w})(\mathbf{u}, p; \mathbf{u}+\delta\bar{\mathbf{v}}, p) \geqslant \min\left\{\frac{1}{2}, \frac{C_\Omega^2}{8}\right\} |||(\mathbf{u}, p)|||^2 \tag{18}$$

On the other hand, using the triangle inequality and the bound for $\bar{\mathbf{v}}$, it can be readily seen that

$$|||(\mathbf{u}+\delta\bar{\mathbf{v}}, p)||| \leqslant \left(1+\frac{C_\Omega}{4}\right)|||(\mathbf{u}, p)||| \tag{19}$$

The assertion now readily follows from (18) and (19). □

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3. ENERGY NORM *A POSTERIORI* ERROR ESTIMATION

In this section, we present a reliable and efficient energy norm error estimator for the exactly divergence-free DG approximations of the Navier–Stokes equation with small data.

3.1. Error indicators

We begin by introducing the error indicators. To that end, let $(\mathbf{u}_h, p_h)$ be the DG approximation of (5). Let $\mathbf{f}_h$ be a piecewise polynomial approximation of $\mathbf{f}$, possibly discontinuous across elemental edges. For any element $K \in \mathcal{T}_h$ and interior edge $E \in \mathcal{E}_I(\mathcal{T}_h)$, we introduce the residuals

$$\mathbf{R}_K = (\mathbf{f}_h + \nu\Delta\mathbf{u}_h - (\mathbf{u}_h \cdot \nabla)\mathbf{u}_h - \nabla p_h)|_K$$

$$\mathbf{R}_E = [\![(p_h \underline{I} - \nu\nabla\mathbf{u}_h)\mathbf{n}]\!]|_E$$

respectively. Here, the matrix $\underline{I}$ is the identity matrix in $\mathbb{R}^{2\times 2}$. For each $K \in \mathcal{T}_h$, we then introduce the local error indicator η_K

$$\eta_K^2 = \eta_{R_K}^2 + \eta_{E_K}^2 + \eta_{J_K}^2 \tag{20}$$

where

$$\begin{aligned} \eta_{R_K}^2 &= \nu^{-1} h_K^2 \|\mathbf{R}_K\|_{L^2(K)}^2 \\ \eta_{E_K}^2 &= \frac{1}{2} \sum_{E \in \partial K \setminus \Gamma} \nu^{-1} h_E \|\mathbf{R}_E\|_{L^2(E)}^2 \\ \eta_{J_K}^2 &= \frac{1}{2} \sum_{E \in \partial K} \kappa_E \nu \|[\![\mathbf{u}_h]\!]\|_{L^2(E)}^2 \end{aligned} \tag{21}$$

Finally, we introduce the data oscillation term

$$\mathbf{O}_K = (\mathbf{f} - \mathbf{f}_h)|_K, \quad K \in \mathcal{T}_h$$

and define

$$\omega_K^2 = \nu^{-1} h_K^2 \|\mathbf{O}_K\|_{L^2(K)}^2 \tag{22}$$

The error estimator η is now given by

$$\eta = \left(\sum_{K \in \mathcal{T}_h} \eta_K^2 \right)^{1/2} \tag{23}$$

whereas the data error ω is defined by

$$\omega = \left(\sum_{K \in \mathcal{T}_h} \omega_K^2 \right)^{1/2} \tag{24}$$

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3.2. Reliability

We now state that the error indicator η gives rise to a reliable energy norm *a posteriori* error estimator, up to the data approximation term ω.

We assume that the data are small and satisfy

$$\max\{1,\gamma^{-1}\}c_0\bar{c}_p v^{-2}\|\mathbf{f}\|_{L^2(\Omega)}\leqslant\tfrac{1}{2} \tag{25}$$

The following result holds.

Theorem 3.1

Assume (25). Let $(\mathbf{u}, p)$ be the solution of the Navier–Stokes equation (1) and $(\mathbf{u}_h, p_h)\in\mathbf{V}_h\times Q_h$ the DG approximation obtained by (5). Let η and ω be the error estimator and the data approximation term in (23) and (24), respectively. Then the following *a posteriori* error bound holds

$$|||(\mathbf{u}-\mathbf{u}_h, p-p_h)|||\leqslant C(\eta+\omega)$$

with a constant $C>0$ that is independent of the viscosity and the mesh size.

The proof of this theorem will be presented in Section 4.1.

Next, we state that the local error indicators can be bounded from above by the local energy error, up to data approximation errors. Hence, the estimator η is also efficient. We first consider the residuals η_{R_K} and η_{E_K}. To that end, we define δ_K by

$$\delta_K=\{K'\in\mathscr{T}_h : K \text{ and } K' \text{ share an edge or a vertex}\}$$

Theorem 3.2

Assume (25). Let $(\mathbf{u}, p)$ be the solution of the Navier–Stokes equation (1) and $(\mathbf{u}_h, p_h)\in\mathbf{V}_h\times Q_h$ the DG approximation obtained by (5). Let η_{R_K}, η_{E_K} and ω_K be defined in (21) and (22), respectively. For any element $K\in\mathscr{T}_h$, there holds

$$\eta_{R_K}\leqslant C(v^{1/2}\|\mathbf{u}-\mathbf{u}_h\|_{H^1(K)}+v^{-1/2}\|p-p_h\|_{L^2(K)}+\omega_K)$$

$$\eta_{E_K}\leqslant C\sum_{K\in\delta_K}(v^{1/2}\|\mathbf{u}-\mathbf{u}_h\|_{H^1(K)}+v^{-1/2}\|p-p_h\|_{L^2(K)}+\omega_K)$$

with a constant $C>0$ that is independent of K, the viscosity and the mesh size.

The proof of this theorem will be given in Sections 4.2 and 4.3.

Finally, let us note that we trivially have efficiency for the jump residuals η_{J_K}. Indeed, since the jumps of the exact solution vanish, there holds

$$\eta_{J_K}^2=\frac{1}{2}\sum_{E\in\partial K\setminus\Gamma}\kappa_E v\|[\![\mathbf{u}-\mathbf{u}_h]\!]\|^2_{L^2(E)}+\sum_{E\in\partial K\cap\Gamma}\kappa_E v\|[\![\mathbf{u}-\mathbf{u}_h]\!]\|^2_{L^2(E)}$$

As a consequence of the above results, we have the following lower bound of the energy error.

Corollary 3.1

Assume (25). Let $(\mathbf{u}, p)$ be the solution of the Navier–Stokes equation (1) and $(\mathbf{u}_h, p_h)\in\mathbf{V}_h\times Q_h$ the DG approximation obtained by (5). Let η and ω be the error estimator and the data approximation term in (23) and (24), respectively. Then the following efficiency bound holds:

$$\eta \leqslant C(|||(\mathbf{u}-\mathbf{u}_h, p-p_h)|||+\omega)$$

with a constant $C>0$ that is independent of the viscosity and the mesh size.

4. PROOFS

In this section, we present the proofs of Theorems 3.1 and 3.2.

4.1. Proof of Theorem 3.1

We introduce the discontinuous RT space $\widetilde{\mathbf{V}}_h = \{\mathbf{v}\in L^2(\Omega)^2 : \mathbf{v}|_K \in \mathrm{RT}_k(K), K\in\mathscr{T}_h\}$. Then, following [9, Section 4], we define $\mathbf{V}_h^c = \widetilde{\mathbf{V}}_h \cap H_0^1(\Omega)^2$. The orthogonal complement of $\mathbf{V}_h^c$ in $\widetilde{\mathbf{V}}_h$ with respect to the norm $\|\cdot\|_{1,h}$ is denoted by $\mathbf{V}_h^{\perp}$. Hence, we have $\widetilde{\mathbf{V}}_h = \mathbf{V}_h^c \oplus \mathbf{V}_h^{\perp}$.

If $\mathbf{u}_h$ is the DG velocity approximation, we can decompose it uniquely into

$$\mathbf{u}_h = \mathbf{u}_h^c + \mathbf{u}_h^r \tag{26}$$

with $\mathbf{u}_h^c \in \mathbf{V}_h^c$ and $\mathbf{u}_h^r \in \mathbf{V}_h^{\perp}$. Then, since $\mathbf{u}_h\in\mathbf{V}_h$ and $\mathbf{u}_h^c\in\mathbf{V}_h^c\subset\mathbf{V}_h$, we must also have that $\mathbf{u}_h^r = \mathbf{u}_h - \mathbf{u}_h^c \in \mathbf{V}_h$. By employing arguments analogous to those in [9, Section 4; 16] for the discontinuous space $\widetilde{\mathbf{V}}_h$, the following result is obtained: there is $c_e>0$, independent of the viscosity and the mesh size, such that

$$\nu^{1/2}\|\mathbf{u}_h^r\|_{1,h} \leqslant c_e \left(\sum_{K\in\mathscr{T}_h} \eta_{J_K}^2\right)^{1/2} \tag{27}$$

Let now $(\mathbf{u}, p)$ be the solution of the Navier–Stokes equations, and $(\mathbf{u}_h, p_h)$ the DG solution. We denote the error of the DG approximation by

$$(\mathbf{e}_u, e_p) = (\mathbf{u}-\mathbf{u}_h, p-p_h)$$

and also set $\mathbf{e}_u^c = \mathbf{u}-\mathbf{u}_h^c$.

Lemma 4.1
Assume (25). Then there is $(\mathbf{v}, q)\in H_0^1(\Omega)^2\times L_0^2(\Omega)$ such that $|||(\mathbf{v},q)|||\leqslant 1$ and

$$\frac{\gamma}{2}|||(\mathbf{e}_u, e_p)||| \leqslant \int_\Omega \mathbf{f}\cdot(\mathbf{v}-\mathbf{v}_h)\,\mathrm{d}\mathbf{x} - \mathscr{A}_h(\mathbf{u}_h)(\mathbf{u}_h, p_h; \mathbf{v}-\mathbf{v}_h, q) + (c_{\mathscr{A}}+\gamma)c_e\left(\sum_{K\in\mathscr{T}_h}\eta_{J_K}^2\right)^{1/2}$$

for any $\mathbf{v}_h\in\mathbf{V}_h$.

Here, note that, as $\mathbf{u}_h$ is exactly divergence free, an approximation q_h of q is not needed (and is set to zero).

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Proof
From the triangle inequality and (27), we have

$$\begin{aligned}\gamma|||(\mathbf{e}_u, e_p)||| &\leqslant \gamma|||(\mathbf{e}_u^c, e_p)||| + \gamma|||(\mathbf{u}_h^r, 0)|||\\ &\leqslant \gamma|||(\mathbf{e}_u^c, e_p)||| + \gamma c_e \left(\sum_{K\in\mathscr{T}_h} \eta_{J_K}^2\right)^{1/2}\end{aligned}$$

Furthermore, from the stability estimate (12), the fact that (25) implies condition (15), the inf–sup condition in Lemma 2.2 is applicable to $(\mathbf{e}_u^c, e_p)$. It follows that there is a test function $(\mathbf{v}, q)\in H_0^1(\Omega)^2\times L_0^2(\Omega)$ such that $|||(\mathbf{v}, q)|||\leqslant 1$ and

$$\begin{aligned}\gamma|||(\mathbf{e}_u^c, e_p)||| &\leqslant \mathscr{A}_h(\mathbf{u}_h)(\mathbf{e}_u^c, e_p; \mathbf{v}, q)\\ &= \mathscr{A}_h(\mathbf{u}_h)(\mathbf{e}_u, e_p; \mathbf{v}, q) + \mathscr{A}_h(\mathbf{u}_h)(\mathbf{u}_h^r, 0; \mathbf{v}, q)\\ &\leqslant \mathscr{A}_h(\mathbf{u}_h)(\mathbf{e}_u, e_p; \mathbf{v}, q) + c_{\mathscr{A}}|||(\mathbf{u}_h^r, 0)|||\\ &\leqslant \mathscr{A}_h(\mathbf{u}_h)(\mathbf{e}_u, e_p; \mathbf{v}, q) + c_{\mathscr{A}} c_e \left(\sum_{K\in\mathscr{T}_h} \eta_{J_K}^2\right)^{1/2}\end{aligned}$$

Here, we have also used the continuity of $\mathscr{A}_h$ in Lemma 2.1 and bound (27). As

$$\mathscr{A}(\mathbf{u}_h)(\mathbf{u}, p; \mathbf{v}, q) = \mathscr{A}_h(\mathbf{u})(\mathbf{u}, p; \mathbf{v}, q) - O_h(\mathbf{e}_u; \mathbf{u}, \mathbf{v})$$

we conclude that

$$\begin{aligned}\gamma|||(\mathbf{e}_u, e_p)||| &\leqslant \mathscr{A}_h(\mathbf{u})(\mathbf{u}, p; \mathbf{v}, q) - O_h(\mathbf{e}_u; \mathbf{u}, \mathbf{v})\\ &\quad - \mathscr{A}_h(\mathbf{u}_h)(\mathbf{u}_h, p_h; \mathbf{v}, q) + (c_{\mathscr{A}} c_e + \gamma c_e)\left(\sum_{K\in\mathscr{T}_h} \eta_{J_K}^2\right)^{1/2}\end{aligned} \tag{28}$$

Owing to the continuity of O_h in (8), the stability bound (2) and the smallness assumption (25), we have

$$\begin{aligned}|O_h(\mathbf{e}_u; \mathbf{u}, \mathbf{v})| &\leqslant c_0\|\mathbf{e}_u\|_{1,h}\|\mathbf{u}\|_{1,h}\|\mathbf{v}\|_{1,h}\\ &\leqslant c_0\bar{c}_p \nu^{-2}\|\mathbf{f}\|_{L^2(\Omega)}\nu^{1/2}\|\mathbf{e}_u\|_{1,h}\nu^{1/2}\|\mathbf{v}\|_{1,h}\\ &\leqslant \frac{\gamma}{2}\nu^{1/2}\|\mathbf{e}_u\|_{1,h}\end{aligned}$$

Hence, this term can be brought to the left-hand side of (28). Then, we have that

$$\mathscr{A}_h(\mathbf{u})(\mathbf{u}, p; \mathbf{v}, q) = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}$$

Therefore, we obtain

$$\frac{\gamma}{2}|||(\mathbf{e}_u, e_p)|||\leqslant \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} - \mathscr{A}_h(\mathbf{u}_h)(\mathbf{u}_h, p_h; \mathbf{v}, q) + (c_{\mathscr{A}} c_e + \gamma c_e)\left(\sum_{K\in\mathscr{T}_h} \eta_{J_K}^2\right)^{1/2}$$

The assertion now follows from this inequality by noting that

$$\mathscr{A}_h(\mathbf{u}_h)(\mathbf{u}_h, p_h; \mathbf{v}_h, 0) - \int_\Omega \mathbf{f}\cdot\mathbf{v}_h \,\mathrm{d}\mathbf{x} = 0$$

for all $\mathbf{v}_h \in \mathbf{V}_h$. □

Lemma 4.2
For $(\mathbf{v}, q) \in H_0^1(\Omega)^2 \times L_0^2(\Omega)$ there is $\mathbf{v}_h \in \mathbf{V}_h$ such that

$$\int_\Omega \mathbf{f}\cdot(\mathbf{v}-\mathbf{v}_h)\,\mathrm{d}\mathbf{x} - \mathscr{A}_h(\mathbf{u}_h)(\mathbf{u}_h, p_h; \mathbf{v}-\mathbf{v}_h, q) \leqslant C\nu^{1/2}\|\nabla\mathbf{v}\|_{L^2(\Omega)}(\eta+\omega)$$

with a constant $C>0$ that is independent of the viscosity and the mesh size.

Proof
We set $\xi_v = \mathbf{v}-\mathbf{v}_h$, with $\mathbf{v}_h \in \mathbf{V}_h^c$ to be selected. Then,

$$\begin{aligned} T &= \int_\Omega \mathbf{f}\cdot\xi_v \,\mathrm{d}\mathbf{x} - \mathscr{A}_h(\mathbf{u}_h)(\mathbf{u}_h, p_h; \xi_v, q) \\ &= \int_\Omega \mathbf{f}\cdot\xi_v \,\mathrm{d}\mathbf{x} - A_h(\mathbf{u}_h, \xi_v) - O_h(\mathbf{u}_h; \mathbf{u}_h, \xi_v) - B_h(\xi_v, p_h) \end{aligned}$$

Here, we have used that $B_h(\mathbf{u}_h, q) = 0$, thanks to the fact that $\mathbf{u}_h$ is divergence free. Integration by parts yields

$$\begin{aligned} -A_h(\mathbf{u}_h, \xi_v) &= -\nu\int_\Omega (\nabla\mathbf{u}_h - \underline{\mathscr{L}}(\mathbf{u}_h)) : \nabla\xi_v \,\mathrm{d}\mathbf{x} \\ &\leqslant \sum_{K\in\mathscr{T}_h}\left(\int_K \nu\Delta\mathbf{u}_h\cdot\xi_v\,\mathrm{d}\mathbf{x} - \int_{\partial K\setminus\Gamma}(\nu\nabla\mathbf{u}_h)\mathbf{n}_K\cdot\xi_v\,\mathrm{d}s\right) \\ &\quad + \nu^{1/2}\|\underline{\mathscr{L}}(\mathbf{u}_h)\|_{L^2(\Omega)}\nu^{1/2}\|\nabla\xi_v\|_{L^2(\Omega)} \end{aligned}$$

with $\mathbf{n}_K$ denoting the unit outward normal vector on ∂K. There holds

$$\nu^{1/2}\|\underline{\mathscr{L}}(\mathbf{u}_h)\|_{L^2(\Omega)} \leqslant c_l\left(\sum_{K\in\mathscr{T}_h}\eta_{J_K}^2\right)^{1/2}$$

for a constant $c_l>0$ that is independent of the viscosity and the mesh size, see [18, Lemmas 7.2 and 7.4]. We further have

$$-B_h(\xi_v, p_h) = -\sum_{K\in\mathscr{T}_h}\left(\int_K \nabla p_h\cdot\xi_v\,\mathrm{d}\mathbf{x} - \sum_{K\setminus\Gamma}\int_{\partial K}(p_h\mathbf{n}_K)\cdot\xi_v\,\mathrm{d}s\right)$$

and

$$-O_h(\mathbf{u}_h; \mathbf{u}_h, \xi_v) = -\sum_{K\in\mathscr{T}_h}\int_K (\mathbf{u}_h\cdot\nabla)\mathbf{u}_h\cdot\xi_v\,\mathrm{d}\mathbf{x}$$

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Combining the above relations, we conclude that

$$T \leqslant \sum_{K\in\mathscr{T}_h}\int_K (\mathbf{R}_K+\mathbf{O}_K)\cdot\boldsymbol{\xi}_v\,\mathrm{d}\mathbf{x}+\sum_{E\in\mathscr{E}_I(\mathscr{T}_h)}\int_E \mathbf{R}_E\cdot\boldsymbol{\xi}_v\,\mathrm{d}s$$
$$+c_l\left(\sum_{K\in\mathscr{T}_h}\eta_{J_K}^2\right)^{1/2} v^{1/2}\|\nabla\boldsymbol{\xi}_v\|_{L^2(\Omega)} \tag{29}$$

We now choose $\mathbf{v}_h\in\mathbf{V}_h^c$ as the standard Scott–Zhang interpolant of $\mathbf{v}$, see, e.g. [25]. It satisfies

$$\sum_{K\in\mathscr{T}_h}(h_K^{-2}\|\mathbf{v}-\mathbf{v}_h\|_{L^2(K)}^2+\|\nabla(\mathbf{v}-\mathbf{v}_h)\|_{L^2(K)}^2)\leqslant C\|\nabla\mathbf{v}\|_{L^2(\Omega)}^2 \tag{30}$$

and

$$\sum_{E\in\mathscr{E}_I(\mathscr{T}_h)} h_E^{-1}\|\mathbf{v}-\mathbf{v}_h\|_{L^2(E)}^2\leqslant C\|\nabla\mathbf{v}\|_{L^2(\Omega)}^2 \tag{31}$$

Note that on an edge with a hanging node we use the interpolant from the unrefined side in order to maintain conformity. The assertion now follows from (29), by using the weighted Cauchy–Schwarz inequality and the approximation properties in (30)–(31). □

Theorem 3.1 is an immediate consequence of Lemmas 4.1 and 4.2.

4.2. Efficiency bound for η_{R_K}

In this section, we prove the efficiency bound for η_{R_K} stated in Theorem 3.2. We do so by employing the bubble function technique introduced in [26, 27].

Let K be an element of $\mathscr{T}_h$. We denote by b_K the standard polynomial bubble function on K. Let now $\mathbf{v}$ be a (vector-valued) polynomial function on K; then there exists a constant $C>0$ independent of $\mathbf{v}$ and K such that

$$\|b_K\mathbf{v}\|_{L^2(K)}\leqslant C\|\mathbf{v}\|_{L^2(K)} \tag{32}$$

$$\|\mathbf{v}\|_{L^2(K)}\leqslant C\|b_K^{1/2}\mathbf{v}\|_{L^2(K)} \tag{33}$$

$$\|\nabla(b_K\mathbf{v})\|_{L^2(K)}\leqslant Ch_K^{-1}\|\mathbf{v}\|_{L^2(K)} \tag{34}$$

$$\|b_K\mathbf{v}\|_{L^\infty(K)}\leqslant Ch_K^{-1}\|\mathbf{v}\|_{L^2(K)} \tag{35}$$

The proof of (32) and (33) is given in [26, Lemma 4.1]. The proof of (34) and (35) can be obtained by similar arguments, see [28, Theorems 2.2 and 2.4].

Fix an element $K\in\mathscr{T}_h$ and set $\mathbf{V}_b=b_K\mathbf{R}_K$. By (33), the definition of the data approximation term $\mathbf{O}_K$, and the fact that $(\mathbf{u}, p)$ solves the Navier–Stokes equations, we have

$$C\|\mathbf{R}_K\|_{L^2(K)}^2 \leqslant \|b_K^{1/2}\mathbf{R}_K\|_{L^2(K)}^2=\int_K \mathbf{R}_K\cdot\mathbf{V}_b\,\mathrm{d}\mathbf{x}$$
$$=\int_K (\mathbf{f}+v\Delta\mathbf{u}_h-(\mathbf{u}_h\cdot\nabla)\mathbf{u}_h-\nabla p_h)\cdot\mathbf{V}_b\,\mathrm{d}\mathbf{x}-\int_K \mathbf{O}_K\cdot\mathbf{V}_b\,\mathrm{d}\mathbf{x}$$

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Hence,

$$C\|\mathbf{R}_K\|_{L^2(K)}^2 \leqslant T_1+T_2+T_3 \tag{36}$$

where

$$T_1=\int_K(-\nu\Delta\mathbf{e}_u+\nabla e_p)\cdot\mathbf{V}_b\,\mathrm{d}\mathbf{x}$$

$$T_2=\int_K((\mathbf{u}\cdot\nabla)\mathbf{u}-(\mathbf{u}_h\cdot\nabla)\mathbf{u}_h)\cdot\mathbf{V}_b\,\mathrm{d}\mathbf{x}$$

$$T_3=-\int_K\mathbf{O}_K\cdot\mathbf{V}_b\,\mathrm{d}\mathbf{x}$$

To bound T_1, we first integrate by parts over K and use the Cauchy–Schwarz inequality:

$$T_1\leqslant C\nu^{1/2}\|\nabla\mathbf{V}_b\|_{L^2(K)}(\nu^{1/2}\|\nabla\mathbf{e}_u\|_{L^2(K)}+\nu^{-1/2}\|e_p\|_{L^2(K)})$$

Upon application of (34), we then obtain

$$T_1\leqslant C\nu^{1/2}h_K^{-1}\|\mathbf{R}_K\|_{L^2(K)}(\nu^{1/2}\|\nabla\mathbf{e}_u\|_{L^2(K)}+\nu^{-1/2}\|e_p\|_{L^2(K)}) \tag{37}$$

For the term T_2, we proceed as follows. By the Cauchy–Schwarz inequality and (35), we have

$$\begin{aligned}T_2&=\int_K((\mathbf{e}_u\cdot\nabla)\mathbf{u}+(\mathbf{u}_h\cdot\nabla)\mathbf{e}_u)\cdot\mathbf{V}_b\,\mathrm{d}\mathbf{x}\\&\leqslant C\|\mathbf{V}_b\|_{L^\infty(K)}\|\mathbf{e}_u\|_{H^1(K)}(\|\nabla\mathbf{u}\|_{L^2(K)}+\|\mathbf{u}_h\|_{L^2(K)})\\&\leqslant Ch_K^{-1}\|\mathbf{R}_K\|_{L^2(K)}\|\mathbf{e}_u\|_{H^1(K)}(\|\nabla\mathbf{u}\|_{L^2(K)}+\|\mathbf{u}_h\|_{L^2(K)})\end{aligned}$$

The stability bound (2) and the smallness assumption (25) then yield

$$\|\nabla\mathbf{u}\|_{L^2(K)}\leqslant\bar{c}_p\nu^{-1}\|\mathbf{f}\|_{L^2(\Omega)}=c_0^{-1}c_0\bar{c}_p\nu^{-1}\|\mathbf{f}\|_{L^2(\Omega)}\leqslant C\nu$$

From the discrete Poincaré inequality (13) and the bound (12), we conclude similarly that

$$\|\mathbf{u}_h\|_{L^2(K)}\leqslant\|\mathbf{u}_h\|_{L^2(\Omega)}\leqslant c_p\|\mathbf{u}_h\|_{1,h}\leqslant c_p\bar{c}_p\nu^{-1}\|\mathbf{f}\|_{L^2(\Omega)}\leqslant C\nu$$

Hence,

$$T_2\leqslant C\nu^{1/2}h_K^{-1}\|\mathbf{R}_K\|_{L^2(K)}\nu^{1/2}\|\mathbf{e}_u\|_{H^1(K)} \tag{38}$$

Finally, for the term T_3, we use (32) and conclude that

$$T_3\leqslant C\|\mathbf{f}-\mathbf{f}_h\|_{L^2(K)}\|\mathbf{V}_b\|_{L^2(K)}\leqslant C\|\mathbf{f}-\mathbf{f}_h\|_{L^2(K)}\|\mathbf{R}_K\|_{L^2(K)} \tag{39}$$

We now combine (36) with the bounds in (37), (38), and (39), divide the resulting estimate by $\|\mathbf{R}_K\|_{L^2(K)}$ and multiply it with $\nu^{-1/2}h_K$. This yields

$$\eta_{R_K}=\nu^{-1/2}h_K\|\mathbf{R}_K\|_{L^2(K)}\leqslant C(\nu^{1/2}\|\mathbf{e}_u\|_{H^1(K)}+\nu^{-1/2}\|e_p\|_{L^2(K)}+\omega_K) \tag{40}$$

This is the desired bound for η_{R_K}.

4.3. Efficiency bound for η_{E_K}

Next, we show the efficiency of η_{E_K} stated in Theorem 3.2.

Consider an interior edge E that is shared by two elements K and K'. We denote by b_E the standard polynomial bubble function on E. Let $\delta_E=\{K,K'\}$. If E is a regular edge, we set $\widetilde{K}=K'$. If one vertex of E is a hanging node, we may assume without loss of generality that E is an entire edge of K. We then denote by $\widetilde{K}\subset K'$ the largest rectangle contained in K' so that E is also an entire edge of $\widetilde{K}$.

Let now $\widetilde{\delta}_E=\{K,\widetilde{K}\}$. If $\mathbf{w}$ is a (vector-valued) polynomial function on E, then there exists a constant $C>0$ independent of $\mathbf{w}$ and h_E such that

$$\|\mathbf{w}\|_{L^2(E)}\leqslant C\|b_E^{1/2}\mathbf{w}\|_{L^2(E)} \tag{41}$$

Furthermore, there exists an extension $\mathbf{w}_b\in H_0^1(K\cap\widetilde{K})^2$ such that $\mathbf{w}_b|_E=b_E\mathbf{w}$ and

$$\|\mathbf{w}_b\|_{L^2(K)}\leqslant Ch_E^{1/2}\|\mathbf{w}\|_{L^2(E)}\quad \forall K\in\widetilde{\delta}_E \tag{42}$$

$$\|\nabla\mathbf{w}_b\|_{L^2(K)}\leqslant Ch_E^{-1/2}\|\mathbf{w}\|_{L^2(E)}\quad \forall K\in\widetilde{\delta}_E \tag{43}$$

$$\|\mathbf{w}_b\|_{L^\infty(K)}\leqslant Ch_E^{-1/2}\|\mathbf{w}\|_{L^2(E)}\quad \forall K\in\widetilde{\delta}_E \tag{44}$$

with a constant $C>0$ independent of $\mathbf{w}$ and E; cf. [26, Lemma 4.1; 28, Theorems 2.2 and 2.4]. We extend $\mathbf{w}_b$ by zero outside the patch formed by the union of K and $\widetilde{K}$.

Let now $\mathbf{R}_E$ be the edge residual over the edge E. We denote by $\mathbf{W}_b$ the extension of $b_E\mathbf{R}_E$ constructed above. By (41), we obtain

$$C\|\mathbf{R}_E\|^2_{L^2(E)}\leqslant\|b_E^{1/2}\mathbf{R}_E\|^2_{L^2(E)}=\int_E\mathbf{R}_E\cdot\mathbf{W}_b\,\mathrm{d}s$$

To estimate the latter integral, we first note that the solution $(\mathbf{u},p)$ of the Navier–Stokes equations satisfies

$$[\![(p\underline{I}-\nu\nabla\mathbf{u})\mathbf{n}]\!]|_E=\mathbf{0}$$

Using this and Green's formula in each element of the patch $\widetilde{\delta}_E$, we then conclude that

$$\int_E\mathbf{R}_E\cdot\mathbf{W}_b\,\mathrm{d}s=\sum_{K\in\widetilde{\delta}_E}\int_K((-\nu\Delta\mathbf{e}_u+\nabla e_p)\cdot\mathbf{W}_b+(e_p\underline{I}-\nu\nabla\mathbf{e}_u):\nabla\mathbf{W}_b)\,\mathrm{d}\mathbf{x}$$

Consequently, using that $(\mathbf{u},p)$ solves the Navier–Stokes equations, we see that

$$\begin{aligned}\int_E\mathbf{R}_E\cdot\mathbf{W}_b\,\mathrm{d}s=&\sum_{K\in\widetilde{\delta}_E}\int_K(\mathbf{f}+\nu\Delta\mathbf{u}_h-\nabla p_h-(\mathbf{u}_h\cdot\nabla)\mathbf{u}_h)\cdot\mathbf{W}_b\,\mathrm{d}\mathbf{x}\\&+\sum_{K\in\widetilde{\delta}_E}\int_K((\mathbf{u}_h\cdot\nabla)\mathbf{u}_h-(\mathbf{u}\cdot\nabla)\mathbf{u})\cdot\mathbf{W}_b\,\mathrm{d}\mathbf{x}\\&+\sum_{K\in\widetilde{\delta}_E}\int_K(e_p\underline{I}-\nu\nabla\mathbf{e}_u):\nabla\mathbf{W}_b\,\mathrm{d}\mathbf{x}\end{aligned}$$

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Therefore,

$$C\|\mathbf{R}_E\|^2_{L^2(E)} \leqslant S_1+S_2+S_3 \tag{45}$$

where

$$S_1=\sum_{K\in\widetilde{\delta}_E}\int_K ((\mathbf{f}_h+\nu\Delta\mathbf{u}_h-(\mathbf{u}_h\cdot\nabla)\mathbf{u}_h-\nabla p_h)+(\mathbf{f}-\mathbf{f}_h))\cdot\mathbf{W}_b\,\mathrm{d}\mathbf{x}$$

$$S_2=\sum_{K\in\widetilde{\delta}_E}\int_K ((\mathbf{u}_h\cdot\nabla)\mathbf{u}_h-(\mathbf{u}\cdot\nabla)\mathbf{u})\cdot\mathbf{W}_b\,\mathrm{d}\mathbf{x}$$

$$S_3=\sum_{K\in\widetilde{\delta}_E}\int_K (e_p\underline{I}-\nu\nabla\mathbf{e}_u):\nabla\mathbf{W}_b\,\mathrm{d}\mathbf{x}$$

To bound S_1, we employ the Cauchy–Schwarz inequality, bound (42), and take into account the shape regularity of the mesh. We obtain

$$\begin{aligned} S_1 &\leqslant C\nu^{1/2}h_E^{-1/2}\|\mathbf{R}_E\|_{L^2(E)}\sum_{K\in\widetilde{\delta}_E}\nu^{-1/2}h_K\|\mathbf{f}_h+\nu\Delta\mathbf{u}_h-(\mathbf{u}_h\cdot\nabla)\mathbf{u}_h-\nabla p_h\|_{L^2(K)} \\ &\quad + C\nu^{1/2}h_E^{-1/2}\|\mathbf{R}_E\|_{L^2(E)}\sum_{K\in\widetilde{\delta}_E}\nu^{-1/2}h_K\|\mathbf{f}-\mathbf{f}_h\|_{L^2(K)} \end{aligned}$$

Obviously, there holds $\|\cdot\|_{L^2(\widetilde{K})} \leqslant \|\cdot\|_{L^2(K')}$. Therefore, we conclude that

$$S_1 \leqslant C\nu^{1/2}h_E^{-1/2}\|\mathbf{R}_E\|_{L^2(E)}\sum_{K\in\delta_E}(\eta_{R_K}+\omega_K) \tag{46}$$

By proceeding as for the bound of T_2 in (38) and using (44), we find that

$$\begin{aligned} S_2 &\leqslant C\nu^{1/2}h_E^{-1/2}\|\mathbf{R}_E\|_{L^2(E)}\sum_{K\in\widetilde{\delta}_E}\nu^{1/2}\|\mathbf{e}_u\|_{H^1(K)} \\ &\leqslant C\nu^{1/2}h_E^{-1/2}\|\mathbf{R}_E\|_{L^2(E)}\sum_{K\in\delta_E}\nu^{1/2}\|\mathbf{e}_u\|_{H^1(K)} \end{aligned} \tag{47}$$

Similarly, employing (43), the sum S_3 can be readily bounded by

$$S_3 \leqslant C\nu^{1/2}h_E^{-1/2}\|\mathbf{R}_E\|_{L^2(E)}\sum_{K\in\delta_E}(\nu^{1/2}\|\nabla\mathbf{e}_u\|_{L^2(K)}+\nu^{-1/2}\|e_p\|_{L^2(\Omega)}) \tag{48}$$

By combining (45)–(48), by dividing the resulting bound by $\|\mathbf{R}_E\|_{L^2(E)}$ and by multiplying it by $\nu^{-1/2}h_E^{1/2}$, we obtain

$$\nu^{-1/2}h_E^{1/2}\|\mathbf{R}_E\|_{L^2(E)} \leqslant C\sum_{K\in\delta_E}(\nu^{1/2}\|\mathbf{e}_u\|_{H^1(K)}+\nu^{-1/2}\|e_p\|_{L^2(K)})+C\sum_{K\in\delta_E}(\eta_{R_K}+\omega_K) \tag{49}$$

The desired bound for η_{E_K} now follows easily from (40) and (49). This concludes the proof of Theorem 3.2.

5. NUMERICAL RESULTS

In this section, we test our error estimate by reproducing known analytical solutions. All the results were obtained using the deal.II finite element library (see [17]). Let us consider the standard singular solution of the Stokes problem with $v=1$ on an L-shaped domain (see [9, 27]). The singularities at the re-entrant corner are of the form r^{λ} and $r^{\lambda-1}$ for the velocity and pressure, respectively, where $\lambda \approx 0.544$. First, we use uniform mesh refinement and do not expect convergence orders better than λ in the energy norm in terms of the number of degrees of freedom n_{dofs}. This is confirmed in Table I for RT_3/Q_3 and RT_2/Q_2 elements. The *a posteriori* error estimate η is of the same order as the actual energy error, thereby confirming the reliability and efficiency results from Theorems 3.1 and 3.2, respectively. The ratio of the estimator to error is also shown and is around 8 for $k=2$ and close to 14 for $k=3$.

In order to use the estimate η for adaptive mesh refinement, we use a heuristic criterion based on cellwise error indicators only. We simply refine the third of the overall cells with the highest indicators. Strategies like this have been used successfully in many applications, see, e.g. [29] and references cited therein. While for this strategy, no quantitative convergence estimates like in [13, 30] are obtained, its implementation is very simple. From the sets η_K and η_E, we obtain cell refinement indicators by adding all or half of the face indicator η_E to its both neighboring cells for boundary and interior faces, respectively. Then, the fraction Θ of the total number of cells is refined.

In Figure 1, we present results on adaptively refined meshes for degrees 2–4, again for the Stokes solution. First, this figure shows that the graphs for the estimates are parallel to those of the error, confirming reliability and efficiency of the estimator. Furthermore, adaptive refinement allows us to recover the optimal convergence rate of $n_{\text{dofs}}^{k/2}$; here, the fraction Θ was chosen as 0.15, 0.1, and 0.05 for degrees 2–4, respectively.

As a second example, we reproduce the analytical Navier–Stokes solution by Kovasznay [31] for a viscosity of $v=\frac{1}{10}$. The estimates and the errors in the norm $|||(\mathbf{u}, p)|||$ are reported in Figure 2. As the solution is smooth, both error graphs are parallel and exhibit first-order convergence. Again, the estimate is by a factor of 5 larger than the error, even if the viscosity is smaller, confirming the robustness of our estimate. On the first mesh, the nonlinearity is unresolved and therefore the estimate cannot capture the error.

The two meshes in Figure 3 show why adaptive mesh refinement yields much better results for the L-shaped domain than for Kovasznay flow: while refinement is concentrated around the re-entrant corner in the former case, it extends to a wide area to the left for the latter.

Table I. Convergence history for the Stokes problem on an L-shaped domain with uniform meshes.

	RT_2/Q_2				RT_3/Q_3			
L	$\|\|\|(\mathbf{e_u}, e_p)\|\|\|$	η	$\frac{\eta}{\|\|\|(\mathbf{e_u}, e_p)\|\|\|}$	Order	$\|\|\|(\mathbf{e_u}, e_p)\|\|\|$	η	$\frac{\eta}{\|\|\|(\mathbf{e_u}, e_p)\|\|\|}$	Order
4	0.612	4.95	8.1	—	0.427	5.87	13.7	—
5	0.420	3.39	8.1	0.54	0.293	4.02	13.7	0.54
6	0.288	2.32	8.1	0.54	0.201	2.76	13.7	0.54
7	0.197	1.59	8.1	0.54	0.138	1.89	13.7	0.54
8	0.135	1.09	8.1	0.54	0.094	1.30	13.7	0.54

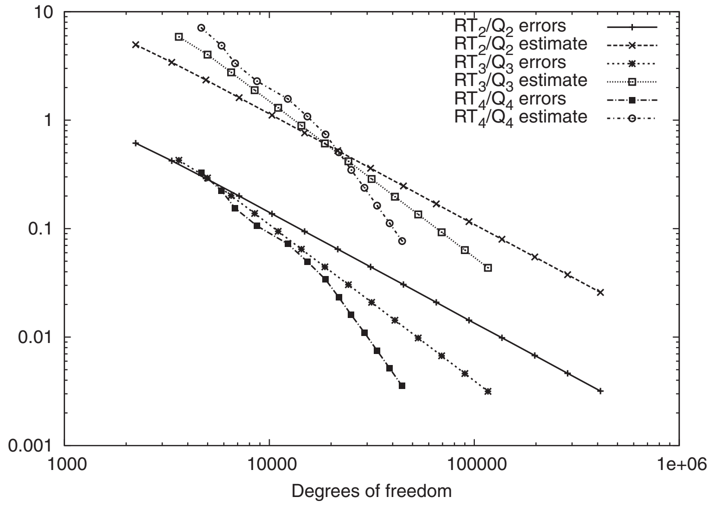

Figure 1. Errors and estimates for adaptive refinement. Stokes problem on an L-shaped domain.

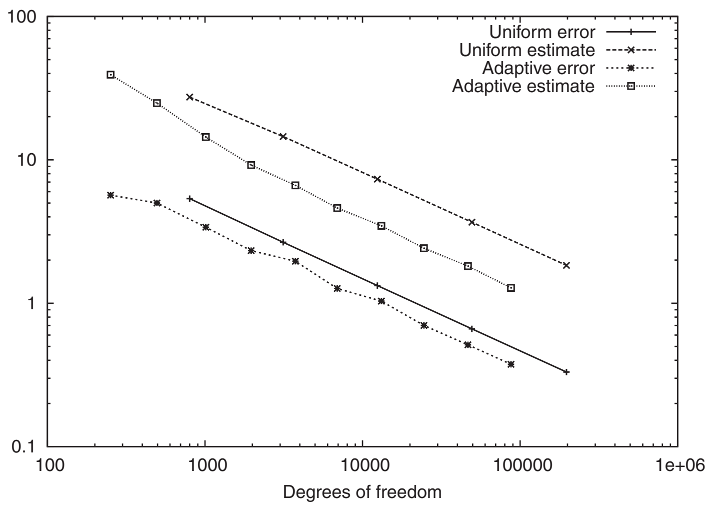

Figure 2. Energy error and estimates for Kovasznay flow, $\nu=0.1$, uniform and adaptive refinement with RT_1/Q_1 elements.

6. CONCLUSION

We presented a residual based *a posteriori* error estimate for DG approximations to solutions of the Navier–Stokes equations. These approximations are based on RT elements on locally refined meshes and yield strongly divergence-free discrete solutions. Therefore, a Helmholtz decomposition of the error is unnecessary and the estimate takes a simple form. The estimate is shown to be

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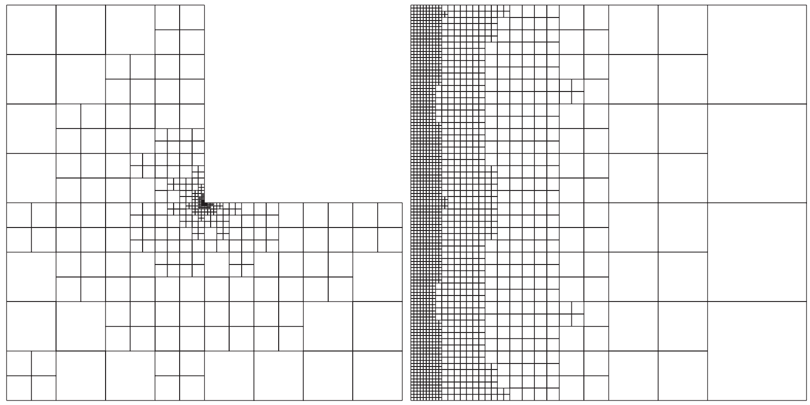

Figure 3. Adaptive meshes for the L-shaped domain (RT_4/Q_4) and for Kovasznay flow (RT_1/Q_1).

reliable, efficient, and robust with respect to the viscosity, as long as the data are small. Numerical results show that the constants involved are of moderate size.

Finally, we point out that our analysis applies naturally to exactly divergence-free BDM_k/P_{k-1} elements on regular triangular meshes; cf. [1].

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